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Journal of Functional Analysis 200 (2003) 281–300

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

Non-commutative shifts and crossed products

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Received 15 June 2000; revised 13 June 2002; accepted 30 September 2002

Communicated by A. Connes

Dedicated to Professor Y. Nakagami on the occasion of his sixtieth birthday

Abstract

When A is a unital simple AF C^*-algebra and has a unique tracial state, it is shown that the crossed product of the two-sided infinite tensor product $\otimes_{\mathbf{Z}}A$ by the shift is a tracially AF C^*-algebra. A similar result is given to the crossed product of a certain *non-unital* two-sided infinite tensor product by the shift. Applying a far-reaching classification result of such C^*-algebras by H. Lin, we obtain an example of a one-parameter automorphism group on some AF C^*-algebra which is not approximately inner, a counter-example to the AF version of the so-called Powers–Sakai conjecture [23].

1. Introduction

By a non-commutative shift we mean the shift automorphism σ of the two-sided infinite (minimal) tensor product $\otimes_{\mathbf{Z}}A$ of copies of some unital (non-commutative) C^*-algebra A. Except for some special cases, the shift σ induces a non-trivial action on K theory; e.g., if the C^*-algebra A is AF, σ_* fixes no non-zero elements of $K_0(\otimes_{\mathbf{Z}}A)$ which are rationally independent of the class $[1]$ of the unit. Our general aim is to study the shift σ; more specifically in this note, the crossed product $(\otimes_{\mathbf{Z}}A)\times_\sigma \mathbf{Z}$, also as a typical example of crossed products by automorphisms inducing non-trivial actions on K theory.

Before going into details we list some results known for non-commutative shifts. (1) If A is completely non-commutative in the sense that A contains a finite-dimensional C^*-subalgebra D such that $1\in D$ and D has no one-dimensional direct

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doi:10.1016/S0022-1236(03)00024-7

summands, then σ has the Rohlin property [17]; a fact which will be often used in this note. (2) If A is a UHF C^*-algebra or a full matrix algebra, then the crossed product $(\otimes_{\mathbf{Z}}A)\times_\sigma \mathbf{Z}$ is a unital simple AT algebra of real rank zero, since the shift is approximately inner (and so induces the identity map on $K_0(\otimes_{\mathbf{Z}}A)$) [7,15,16]. (3) If A is a unital prime AF C^*-algebra (or a full matrix algebra), then the pure σ-invariant states are dense in the σ-invariant states of $\otimes_{\mathbf{Z}}A$ [12,6]. (Only this property depends on σ itself; all the other properties above and below depend just on its outer conjugacy class.) We will also state the following facts. (4) If A is a unital AF C^*-algebra, then $K_i(\otimes_{\mathbf{Z}}A\times_\sigma \mathbf{Z})$ is a torsion-free abelian group for $i=0,1$. (5) If A is a finite-dimensional C^*-algebra with non-trivial center, then $\otimes_{\mathbf{Z}}A\times_\sigma \mathbf{Z}$ is not an AT C^*-algebra. But we should note that this is certainly AF embeddable (i.e., isomorphic to a C^*-subalgebra of an AF C^*-algebra); in general if A is AF embeddable, then $\otimes_{\mathbf{Z}}A\times_\sigma \mathbf{Z}$ is also AF embeddable (cf. [8]).

What we can add in this note is (6) if A is a unital simple AF C^*-algebra with a unique tracial state, then the crossed product $(\otimes_{\mathbf{Z}}A)\times_\sigma \mathbf{Z}$ is a unital simple tracially AF C^*-algebra. This is an attempt to extend the result (2) above but we are still short of showing that $(\otimes_{\mathbf{Z}}A)\times_\sigma \mathbf{Z}$ is an AT C^*-algebra.

The reason why we can prove the above assertion (6) is that if the unital simple AF C^*-algebra has a unique tracial state, then it is *tracially* UHF (see Lemma 2.4); anyway such a C^*-algebra is not very far from being UHF; but the fact of not being UHF also reflects on the conclusion in comparison with (2) above.

In the second part of this note we consider a non-unital version of the above result. This means that we specify two projections $e_-,e_+\in A$ and we use e_- (resp. e_+) as a unit as we move to the left (resp. right); i.e., setting an embedding $\otimes_{-n}^{n}A\to\otimes_{-n-1}^{n+1}A$ as $x\mapsto e_-\otimes x\otimes e_+$ for $n=1,2,\ldots,$ we define the inductive limit C^*-algebra $\otimes_{\mathbf{Z}}(A,e_-,e_+)$, on which the shift σ is well-defined. In this setting we show the following: (7) If A is an AF C^*-algebra and $[e_-]\neq[e_+]$ in $K_0(A)$, then $K_1(\otimes_{\mathbf{Z}}(A,e_-,e_+)\times_\sigma \mathbf{Z})=\{0\}$. (8) If A is an AF C^*-algebra and $[e_-]\neq[e_+]$ and if $p|\,[e_-]-[e_+]$ then both $p|\,[e_-]$ and $p|\,[e_+]$ for all prime numbers p, then $K_0(\otimes_{\mathbf{Z}}(A,e_-,e_+)\times_\sigma \mathbf{Z})$ is a torsion-free abelian group. (9) If A is a unital simple AF C^*-algebra with a unique tracial state τ and $\tau(e_-)=\tau(e_+)$, then $\otimes_{\mathbf{Z}}(A,e_-,e_+)\times_\sigma \mathbf{Z}$ is a tracially AF C^*-algebra. We should hastily add that if $\tau(e_-)\neq\tau(e_+)$, then $\otimes_{\mathbf{Z}}(A,e_-,e_+)\times_\sigma \mathbf{Z}$ is purely infinite [13,14,22,24]; so only the finite case $\tau(e_-)=\tau(e_+)$ has remained for clarification.

What I was hoping for was to prove that $\otimes_{\mathbf{Z}}(A,e_-,e_+)\times_\sigma \mathbf{Z}$ is an AF C^*-algebra in the situation of (9) with e_-,e_+ satisfying the condition in (8). If successful, this would give an example of an action of $\mathbf{T}$ on some unital simple AF C^*-algebra whose fixed point algebra is again simple, a counter-example to the (AF version of) Powers–Sakai conjecture [23,25] (which says that any strongly continuous one-parameter automorphism group of an AF C^*-algebra is approximately inner; in particular that in the periodic case the fixed point algebra is not simple). This way of constructing possible counter-examples to this conjecture had perhaps been known (cf. [3–5]) but it became plausible only after Elliott's paper [11] appeared—I owe this point to A. Kumjian.

In a recent preprint H. Lin [20] has shown a remarkable result: If B is a unital separable nuclear simple tracially AF C^*-algebra and satisfies the Universal Coefficient Theorem, then B is determined by its K theoretic data $(K_0(A), K_0(A)_+, [1_A], K_1(A))$. (Earlier than this, he defined a notion of tracial topological rank; tracially AF C^*-algebras are those of tracial topological rank zero.) Note that unital simple AF C^*-algebras are in this class. Hence in the situation of (9) with (8), it follows that $B = (\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z})$ is an AF C^*-algebra. If e is a non-zero projection in $\otimes_{\mathbf{Z}}(A, e_-, e_+)$, then $D = (e \otimes 1)B(e \otimes 1)$ is a unital AF C^*-algebra which is left invariant under the dual action of σ. If we denote by α the restriction of this action to D, we have the following situation: D is a unital simple AF C^*-algebra, α is an action of $\mathbf{T}$ on D, and the fixed point algebra D^α is simple (being isomorphic to $e(\otimes_{\mathbf{Z}}(A, e_-, e_+))e$). Thus relying on Lin's result [20], we obtain a counter-example to the (AF version of) Powers–Sakai conjecture, i.e. α, as a one-parameter automorphism group of D, is not approximately inner. (Since D cannot be a UHF C^*-algebra, this does not give a counter-example to the original Powers–Sakai conjecture.) Note also that if E is a unital C^*-algebra and β is a strongly continuous one-parameter automorphism group of E, then $\alpha \otimes \beta$ is never approximately inner on $B \otimes E$. (Because if it were, $\alpha \otimes \beta$ should have a ground state, which would give a ground state for α.) In this way we could get more examples (see [25]). Formally we state:

Theorem 1.1. *There exists a unital simple* AF *C^*-algebra A and a strongly continuous one-parameter automorphism group α of A such that α is not approximately inner.*

Before concluding Introduction we will explain some definitions used above and to be used below. A (separable) C^*-algebra A is called AF (approximately finite-dimensional) if there is an increasing sequence (A_n) of C^*-subalgebras of A with $A = \overline{\cup_n A_n}$ such that $\dim A_n < \infty$ for all n; and is called AT [11] if we replace the condition $\dim A_n < \infty$ by $A_n \cong B_n \otimes C(\mathbf{T})$ with $\dim B_n < \infty$ in the above definition. We say that a unital (separable) C^*-algebra A has real rank zero [9] if the invertible elements are dense in the self-adjoint part $A_{sa} = \{x \in A \mid x^* = x\}$ and that A is approximately divisible [2] if A has a central sequence (D_n) of C^*-subalgebras such that $1 \in D_n$, $\dim D_n < \infty$, and D_n has no one-dimensional direct summands for all n. If A is a unital approximately divisible exact C^*-algebra with a unique tracial state, then A has real rank zero [2]. If A is not unital, A has real rank zero if A has an approximate unit (p_n) consisting of projections such that $p_n A p_n$ has real rank zero. (This is equivalent to saying that $A + \mathrm{C}1$ has real rank zero.)

The notion of *tracially* AF was introduced by H. Lin. A unital simple C^*-algebra A is called tracially AF (approximately finite-dimensional) (or, more recently, of tracial topological rank zero) if for any finite subset $\mathscr{F}$ of A, any $\varepsilon > 0$, and any non-zero $q \in A_+$, there is a non-zero projection $p \in A$ and a finite-dimensional C^*-subalgebra D of pAp with $p \in D$ such that

1. $\forall a \in \mathscr{F} \;\; \|[p, a]\| < \varepsilon$,
2. $\forall a \in \mathscr{F} \; \mathrm{dist}(pap, D) < \varepsilon$,
3. $\exists u \in \mathscr{U}(A) \;\; u(1-p)u^* \in \overline{qAq}$,

where $\mathscr{U}(A)$ is the unitary group of A (see 1.2 of [18] and also [10,19,20]). (If furthermore D can be chosen to be a full matrix algebra, then such a C^*-algebra may be called *tracially* UHF.) Since in our case the unital simple C^*-algebra A is exact, approximately divisible, and has a unique tracial state, A is tracially AF if for any finite subset $\mathscr{F}$ of A and any $\varepsilon > 0$ there is a projection $p \in A$ and a finite-dimensional C^*-subalgebra D of pAp with $p \in D$ such that

1. $\forall a \in \mathscr{F} \, \|[p, a]\| < \varepsilon$,
2. $\forall a \in \mathscr{F} \, \mathrm{dist}(pap, D) < \varepsilon$,
3. $\tau(p) > 1 - \varepsilon$,

where τ is the unique tracial state of A (see, e.g., [21]).

Lin [18] shows that if A is a unital simple tracially AF C^*-algebra, then A has real rank zero and stable rank one (though we will not need this fact).

If A is a non-unital simple C^*-algebra, A is tracially AF if A has an approximate unit (p_i) consisting of projections such that $p_i A p_i$ is tracially AF for each i.

2. Unital tensor products

Let A be a unital AF C^*-algebra. We denote by $\otimes_{\mathbf{Z}} A$ the infinite tensor product C^*-algebra $\otimes_{i \in \mathbf{Z}} A(i)$ with $A(i) \equiv A$ and by σ the automorphism of $\otimes_{i \in \mathbf{Z}} A(i)$ sending $x \in A(i)$ to $x \in A(i+1)$; σ will be called the shift (automorphism) of $\otimes_{\mathbf{Z}} A$. Note that $K_0(\otimes_{\mathbf{Z}} A) = \cup_n \otimes_{-n}^{n} K_0(A)$, where $\otimes_{-n}^{n} K_0(A)$ is embedded into $\otimes_{-n-1}^{n+1} K_0(A)$ by $g \mapsto [1] \otimes g \otimes [1]$.

Proposition 2.1. *If A is a unital* AF *C^*-algebra, then $K_i(\otimes_{\mathbf{Z}} A \times_\sigma \mathbf{Z})$ is a torsion-free abelian group for $i = 0, 1$. Moreover $K_1(\otimes_{\mathbf{Z}} A \times_\sigma \mathbf{Z})$ is isomorphic to the subfield of* $\mathbf{Q}$ *generated by $\{n \in \mathbf{N};\ [1] = ng$ for some $g \in K_0(A)\}$.*

Proof. Let F denote the subfield of $\mathbf{Q}$ generated by $\{n \in \mathbf{N};\ [1] = ng$ for some $g \in K_0(A)\}$. Then it follows that $K_0(\otimes_{\mathbf{Z}} A)$ is a module over F and that σ_* is a module homomorphism. Since by the Pimsner–Voiculescu exact sequence $K_1(\otimes_{\mathbf{Z}} A \times_\sigma \mathbf{Z})$ is the kernel of $\mathrm{id} - \sigma_*$ on $K_0(\otimes_{\mathbf{Z}} A)$, it suffices to show that if $g = \sigma_*(g)$, then $g \in F[1]$. Since $g = \sigma_*^k(g)$ for any $k \in \mathbf{N}$, this follows from the fact that if G is a torsion-free abelian group and $h \otimes g = g \otimes h$ in $G \otimes G$, then h and g are rationally dependent.

By the Pimsner–Voiculescu exact sequence again $K_0(\otimes_{\mathbf{Z}} A \times_\sigma \mathbf{Z})$ is isomorphic to $K_0(\otimes_{\mathbf{Z}} A)/\mathrm{Range}(\mathrm{id} - \sigma_*)$, which we have to show is torsion-free. Suppose that $g - \sigma_*(g) = nh$ for some non-zero $g, h \in K_0(\otimes_{\mathbf{Z}} A)$ and $n = 2, 3, \ldots$. We have to show that g is of the form $ng' + a[1]$ with $a \in F$. We may suppose that no prime factors of n appear in F. Since $g - \sigma_*^k(g) = n(h + \sigma_*(h) + \cdots + \sigma_*^{k-1}(h))$ for any $k \in \mathbf{N}$, the problem can be stated as follows: If $G = K_0(A)^{\otimes m}$ and $g \otimes [1] - [1] \otimes g = nh$ in $G \otimes G$ for $g \in G$ and $h \in G \otimes G$ and no prime factors of n divide $[1]$, then show that g is of the form $ng' + a[1]$. To show this we may suppose that $G = \mathbf{Z}^k$ for some

$k>1$ and that $g=(g_i)$ and $[1]=(\xi_i)$. Let p be a prime number which divides n and let s denote the maximum integer such that p^s divides n. Suppose that there is an i such that p^s does not divide g_i. Since $g_i[1]-\xi_i g=nh'$ for some $h'\in G$, p^s does not divide ξ_i (otherwise p would divide $[1]$). The greatest common divisor of ξ_i and p^s is p^t with some $0\leqslant t<s$. Note that p^t divides g_i. Since there exist $b,c\in\mathbf{Z}$ such that $b\xi_i+cp^s=p^t$, it follows that

$$p^t g = cp^s g + b\xi_i g = cp^s g + b(g_i[1]-nh') = p^s(cg-b(p^{-s}n)h') + bg_i[1]$$

and hence that $g=p^{s-t}g'+b(p^{-t}g_i)[1]$ with $g'=cg-b(p^{-s}n)h'$. Since $g'-\sigma_*(g')=(p^{-s+t}n)h$ and $p^{-s+t}n<n$, we can repeat this process for a finite number of times to get the conclusion. $\square$

Proposition 2.2. *If A is a finite-dimensional C^*-algebra with non-trivial center, then $(\otimes_{\mathbf{Z}}A)\times_\sigma\mathbf{Z}$ is not an* AT *C^*-algebra.*

Proof. If $\otimes_{\mathbf{Z}}A\times_\sigma\mathbf{Z}$ were an AT algebra, then any quotient would be an AT algebra. If we denote by Λ the maximal ideal space of A (which is a finite set), then the center of $\otimes_{\mathbf{Z}}A$ is identified with $C(\Pi_{\mathbf{Z}}\Lambda)$. We take two distinct points $p_-,p_+\in\Lambda$ and define a point $x\in\Pi_{\mathbf{Z}}\Lambda$ by $x_n=p_-$ for $n<0$ and $x_n=p_+$ for $n\geqslant 0$. Let Ω be the translation invariant closed subset of $\Pi_{\mathbf{Z}}\Lambda$ generated by x. Then it follows that Ω consists of the translates of x and two limit points (p_-), (p_+), i.e., Ω is homeomorphic to $\{-\infty\}\cup\mathbf{Z}\cup\{+\infty\}$. Let I denote the ideal of $\otimes_{\mathbf{Z}}A$ generated by $C(\Omega^c)$, which is σ-invariant, and E_+ denote the characteristic function of $\{1,2,\ldots\}\cup\{+\infty\}$. Then one can see that $U_\sigma E_+U_\sigma^*$ is a proper subprojection of E_+ in the quotient $\otimes_{\mathbf{Z}}A/I\times_\sigma\mathbf{Z}$ of $\otimes_{\mathbf{Z}}A\times_\sigma\mathbf{Z}$, where U_σ is the unitary implementing σ. Since this quotient contains a proper isometry $U_\sigma E_+ + 1 - E_+$, it cannot be an AT algebra. $\square$

Theorem 2.3. *Let A be a unital simple* AF *C^*-algebra with a unique tracial state and let σ denote the shift automorphism of $\otimes_{\mathbf{Z}}A$. Then the crossed product $(\otimes_{\mathbf{Z}}A)\times_\sigma\mathbf{Z}$ is a unital simple tracially* AF *C^*-algebra with a unique tracial state.*

If A is an AF C^*-algebra, we denote by $T(A)$ the tracial state space of A. There is a natural order-preserving homomorphism φ of $K_0(A)$ into $\mathrm{Aff}(T(A))$, the real affine continuous functions on $T(A)$, and if $g\in K_0(A)$ and $\varphi(g)>0$ (or $\varphi(g)$ is strictly positive on $T(A)$), then $g>0$ (or g is positive and non-zero). We introduce an order on $\mathbf{R}\otimes K_0(A)$ by: $g>0$ if $\varphi(g)>0$ for $g\in\mathbf{R}\otimes K_0(A)$. If $g\in K_0(A)$ and $g>0$ in $\mathbf{R}\otimes K_0(A)$, then $g>0$ in $K_0(A)$.

Lemma 2.4. *Let A be a unital simple* AF *C^*-algebra with a unique tracial state and let (A_n) be an increasing sequence of finite-dimensional C^*-subalgebras of A with $1\in A_1$ and $A=\overline{\cup_n A_n}$. For any $\varepsilon>0$ there exist a $k\in\mathbf{N}$, a projection $p\in A_k\cap A_1'$, and a full matrix C^*-subalgebra D of pA_kp such that $D\supset A_1p$ and $[p]>(1-\varepsilon)[1]$ in $\mathbf{R}\otimes K_0(A_k\cap A_1')$.*

Proof. From the inductive system $A_1 \to A_2 \to \cdots$, we obtain the inductive system of dimension groups $K_0(A_1) \to K_0(A_2) \to \cdots$. By identifying $K_n(A_n)$ with $\mathbf{Z}^{k_n}$, where k_n is the dimension of the center of A_n, we obtain the k_{n+1} by k_n matrix χ_n with non-negative integer components such that $K_0(A_n) \to K_0(A_{n+1})$ is given by the multiplication of χ_n. For $m<n$ we let χ_{mn} denote $\chi_{n-1}\chi_{n-2}\cdots\chi_m$, which gives the map $K_0(A_m) \to K_0(A_n)$. Let ξ_n be the element of $\mathbf{Z}^{k_n}$ corresponding to $1 \in A_n$. By the assumption that A has a unique tracial state, say τ, it follows that for any $i_n \in \{1, 2, \dots, k_n\}$,

$$\left(\frac{\chi_{1n}(i_n,j)\xi_1(j)}{\xi_n(i_n)}\right)_j$$

converges to a fixed vector in $\mathbf{R}^{k_1}$ as $n \to \infty$. Denoting by (p_{ni}) the minimal central projections of A_n, indexed according to the indexing of $\mathbf{Z}^{k_n}$, we note that $\xi_n(i) = \mathrm{rank}(p_{ni})$ in A_n and that the above fixed vector is $(\tau(p_{11}), \tau(p_{12}), \dots, \tau(p_{1k_1})) \equiv (\mu_j)$. We notice that $\mu_j > 0$ and $\sum_j \mu_j = 1$.

Hence for any $\varepsilon > 0$ there exists a $k \in \mathbf{N}$ such that for any $n \in \mathbf{N}$ with $n \geqslant k$, any $i = 1, 2, \dots, k_n$, and any $j = 1, \dots, k_1$,

$$\left|\frac{\chi_{1n}(i,j)}{\xi_n(i)} - \frac{\mu_j}{\xi_1(j)}\right| < \varepsilon.$$

First we find a rational c_j/d_j ($c_j, d_j \in \mathbf{N}$) such that $c_j \gg 1$ and

$$\frac{c_j}{d_j} < \frac{\mu_j}{\xi_1(j)} < \frac{c_j+1}{d_j}.$$

Second we find an $n \in \mathbf{N}$ such that

$$0 \leqslant \frac{\chi_{1n}(i,j)}{\xi_n(i)} - \frac{c_j}{d_j} < \frac{1}{d_j}.$$

Setting d to be the least common multiple of $d_1, d_2, \dots, d_{k_1}$, we express $\xi_n(i)$ as $a_i d + r_i$ with $a_i \in \mathbf{N}$ and $0 \leqslant r_i < d$ and deduce that

$$0 \leqslant \chi_{1n}(i,j) - a_i c_j d/d_j < a_i d/d_j + r_i(c_j+1)/d_j \equiv b_{ij}.$$

We note that

$$\frac{b_{ij}}{\chi_{1n}(i,j)} < \frac{1}{d_j}\left(1 + \frac{c_j+1}{a_i}\right)\frac{\xi_n(i)}{\chi_{1n}(i,j)}.$$

As $n \to \infty$, the right-hand side gets smaller than $1/c_j$, which we can assume is arbitrarily small. Set $b_j = c_j d/d_j \in \mathbf{N}$. Then, since $\mathrm{rank}(p_{ni}p_{1j}) = \chi_{1n}(i,j) \geqslant a_i b_j$ in $A_n \cap A_1'$, we find a projection $p \in A_n \cap A_1'$ such that $[p]$ corresponds to $(a_i b_j)$ in $K_0(A_n \cap A_1') = \oplus K_0(p_{ni}p_{1j}A_n \cap A_1')$. Since $[1] = (\chi_{1n}(i,j))$ in $K_0(A_n \cap A_1')$, we obtain the estimate $[p] > (1-\varepsilon)[1]$. Finally we find a full matrix C^*-subalgebra D of pA_np of the order $\sum_j b_j\xi_1(j)$ such that $D \supset A_1p$. Note that $pA_np \cong \oplus_{i=1}^{k_n} D \otimes M_{a_i}$, where M_a is the a by a matrix algebra. $\square$

Lemma 2.5. *Let A be a unital simple* AF *C^*-algebra with a unique tracial state and let (A_n) be as before. For any $n \in \mathbf{N}$ and $\varepsilon > 0$ there exist a $k \in \mathbf{N}$, a projection $e \in \otimes_{\mathbf{Z}}(A_k \cap A_1')$, a full matrix C^*-subalgebra D of $\otimes_{\mathbf{Z}} A_k$, and a projection F and a unitary V in $(\otimes_{\mathbf{Z}} A_k) \times_\sigma \mathbf{Z}$ such that $1_D \in \otimes_{\mathbf{Z}}(A_k \cap A_1')$, $F, e \in D'$, $1_D \geqslant F \geqslant e$, $D \supset (\otimes_{-n}^n A_1)1_D$, $\|V - 1\| < \varepsilon$, $\mathrm{Ad}\, VU_\sigma(DF) = DF$, and $[e] > (1-\varepsilon)[1]$ in $\mathbf{R} \otimes K_0(\otimes_{\mathbf{Z}}(A_k \cap A_1'))$.*

Proof. By the previous lemma we find a $k \in \mathbf{N}$, a projection $p \in A_k \cap A_1'$, and a full matrix C^*-subalgebra D of pA_kp such that $D \supset A_1p$. By changing indices we may suppose that $k = 2$ and that $A_m \cap A_{m-1}'$ has no one-dimensional direct summands for any m. (As $D \subset A$, this D is not intended to be the D in the statement.)

Let $k \in \mathbf{N}$ be such that $k \gg 1$. Let $w \in \otimes_{\mathbf{Z}}(A_2 \cap \{p\}')$ be a unitary such that $w(1 - \otimes_{-n-k}^{n+1} p) = (1 - \otimes_{-n-k}^{n+1} p)$, $w(\otimes_{-n-k}^{n+1} p) \in D \otimes (\otimes_{-n-k+1}^{n} p) \otimes D$, and $\mathrm{Ad}\, w | \otimes_{-n-k}^{n+1} D$ switches $x \in D$ at $-n-k$ and $x \in D$ at $n+1$. This is possible because D is a full matrix algebra. Hence $\mathrm{Ad}\, w\sigma$ simulates the cyclical permutation, say σ', of the factors of $\otimes_{-n-k}^{n} D$ in the sense that $\mathrm{Ad}\, w\sigma(p \otimes x) = \sigma'(x) \otimes p$ for $x \in \otimes_{-n-k}^{n} D$. In particular we have that $(\mathrm{Ad}\, w\sigma)^{2n+k+1}((\otimes_{-3n-2k-1}^{-n-k-1} p) \otimes x) = x \otimes (\otimes_{n+k+1}^{3n+2k+1} p)$ for $x \in \otimes_{-n-k}^{n} D$.

By using the Rohlin property for σ on $\otimes_{\mathbf{Z}}(A_3 \cap A_2')$ [17], we obtain, for any $\varepsilon > 0$ and $\ell \in \mathbf{N}$, a set of Rohlin towers (e_{ij}); e_{ij}'s are projections for $i = 0, 1$, $j = 0, 1, \ldots, \ell - i$ such that

$$\sum_i \sum_j e_{ij} = 1,$$
$$\|\sigma(e_{ij}) - e_{ij+1}\| < \varepsilon.$$

We can then construct a unitary $u \in \otimes_{\mathbf{Z}} A_3$ such that $\|w - u^*\sigma(u)\| \sim 1/\ell$, by using (e_{ij}) and short continuous paths from $w_{\ell+1}, w_\ell$ to 1 applied by σ^j, $0 \leqslant j \leqslant \ell$, where w_j's are the unitaries defined by $w_0 = 1$, $w_j = w\sigma(w_{j-1})$. By setting $\ell = [(k-1)/2]$ and choosing the paths in $\otimes_{-n-k}^{-n-k+\ell}(A_2 \cap \{p\}') \otimes (\otimes_{-n-k+\ell+1}^{n}\{1, p\}'') \otimes (\otimes_{n+1}^{n+1+\ell} (A_2 \cap \{p\}'))$, we thus obtain a unitary $u \in \otimes_{\mathbf{Z}} A_3$ such that $\|w - u^*\sigma(u)\| \sim 1/k$ and u belongs to the C^*-subalgebra generated by $\otimes_{\mathbf{Z}} A_3 \cap A_2'$ and $\otimes_{-n-k}^{-n-2}(A_2 \cap \{p\}') \otimes (\otimes_{-n-1}^{n}\{1, p\}'') \otimes \otimes_{n+1}^{n+k-1}(A_2 \cap \{p\}')$. Note that u commutes with $\otimes_{-n-1}^{n} A_1$ and p at any $i \in \mathbf{Z}$.

Let

$$D_1 = \mathrm{Ad}\, u(\otimes_{-n-k}^{n} D \otimes \otimes_{n+1}^{n+k} p).$$

Then D_1 is a full matrix algebra containing $(\otimes_{-n-k}^{-n-1} p) \otimes (\otimes_{-n}^{n} D) \otimes (\otimes_{n+1}^{n+k} p)$ and its identity 1_{D_1} equals $\otimes_{-n-k}^{n+k} p$. (This D_1 will be D in the statement.) We let

$v = uw\sigma(u^*)$, which is a unitary satisfying $\|v-1\| \sim 1/k$ and v commutes with p at any point. Since $\mathrm{Ad}\, w\sigma = \mathrm{Ad}\, u^*\, \mathrm{Ad}\, v\sigma\, \mathrm{Ad}\, u$, $\mathrm{Ad}\, v\sigma$ leaves D_1 invariant in the sense that

$$\mathrm{Ad}\, v\sigma(p_{-n-k-1}D_1) = D_1 p_{n+k+1},$$

where p_m means $p \in A$ at $m \in \mathbf{Z}$. Since $\mathrm{Ad}\, v\sigma$ has period $2n+k+1$ in a sense, we obtain a unitary $z \in D_1$ such that $\mathrm{Ad}\, v\sigma(p_{-n-k-1}x) = \mathrm{Ad}\, z(x)p_{n+k+1}$ for $x \in D_1$ and $z^{2n+k+1} = 1$ by regarding z as a unitary in $\otimes_{\mathbf{Z}} A_3$ by adding $1 - \otimes_{-n-k}^{n+k} p$. Note that

$$\mathrm{Ad}\, z^* v U_\sigma(p_{-n-k-1}x) = p_{n+k+1}x, \quad x \in D_1,$$

where U_σ is the canonical unitary implementing σ in $\otimes_{\mathbf{Z}} A \times_\sigma \mathbf{Z}$.

Let $\ell_1, \ell_2 \in \mathbf{N}$ be such that $\ell_1 \gg \ell_2 \gg 1$ and let $\ell = \ell_1 + \ell_2$. By the Rohlin property on $\otimes_{\mathbf{Z}}(A_4 \cap A_3')$ there exists an orthogonal family $(f_i)_{i=-\ell}^{\ell}$ of projections in $\otimes_{\mathbf{Z}}(A_4 \cap A_3')$ and a unitary $v_1 \in \otimes_{\mathbf{Z}}(A_4 \cap A_3')$ such that $v_1 \approx 1$, $\mathrm{Ad}\, v_1\sigma(f_i) = f_{i+1}$, and $[f_i] > 1/(2\ell+2)[1]$ in $\mathbf{R} \otimes K_0(\otimes_{\mathbf{Z}}(A_4 \cap A_3'))$ (see 2.11 of [17]). We define, for $i = -\ell, -\ell+1, \ldots, \ell$,

$$F_i = f_i(\otimes_{-n-k-\ell+i}^{n+k+\ell+i} p).$$

Then $(F_i)_{i=-\ell}^{\ell}$ is an orthogonal family of projections in $\otimes_{\mathbf{Z}} A_4 \cap A_1'$ such that $\mathrm{Ad}\, v_1\sigma(F_i) = F_{i+1}$. Since $[p] > (1-\varepsilon)[1]$ in $\mathbf{R} \otimes K_0(A_2 \cap A_1')$, $[(\otimes_{-n-k-\ell+i}^{n+k+\ell+i} p)]$ is greater than $(1-\varepsilon)^{2n+2k+2\ell+1}[1]$ in $\mathbf{R} \otimes K_0(\otimes_{\mathbf{Z}} A_2 \cap A_1')$. Since ε can be made small independently of n, k, ℓ_1, ℓ_2, we can assume that ε is so small that we still have that $[F_i] > 1/(2\ell+2)[1]$ in $\mathbf{R} \otimes K_0(\otimes_{\mathbf{Z}} A_4 \cap A_1')$.

Note that z and v commute with $\otimes_{-n-k-\ell+i}^{n+k+\ell+i} p$ and f_i, and that $\mathrm{Ad}\, v_1 z^* v U_\sigma(F_i) = F_{i+1}$. We define an almost $\mathrm{Ad}\, v_1 z^* v U_\sigma$-invariant projection $F \in (\otimes_{\mathbf{Z}} A_4) \times_\sigma \mathbf{Z}$ as follows (cf. [15]):

$$F = \sum_{i=-\ell_1}^{\ell_1} F_i + \sum_{i=1}^{\ell_2-1} \frac{i}{\ell_2}(F_{-\ell+i} + F_{\ell-i}) + \sum_{i=1}^{\ell_2-1} \frac{\sqrt{i(\ell_2-i)}}{\ell_2}\Big((v_1 z^* v U_\sigma)^{2\ell_1+\ell_2} F_{-\ell+i} + F_{-\ell+i}(v_1 z^* v U_\sigma)^{2\ell_1+\ell_2}\Big).$$

We can see that F is indeed a projection and that

$$\mathrm{Ad}\, v_1 z^* v U_\sigma(F) - F \sim 1/\sqrt{\ell_2},$$

$$F \geqslant \sum_{i=-\ell_1}^{\ell_1} F_i \equiv e,$$

$$F \leqslant \otimes_{-n-k-1}^{n+k+1} p \leqslant 1_{D_1}.$$

Here we note that e is a projection in $(\otimes_{\mathbf{Z}}A_4 \cap A_1') \cap D_1'$ and that $[e]$ can be assumed to be close to $[1]$ in $\mathbf{R} \otimes K_0(\otimes_{\mathbf{Z}}A_4 \cap A_1')$ since $\ell_1 \gg \ell_2$. For $x \in D_1$ and $i = 1, 2, \ldots, \ell_2 - 1$ we compute:

$$\begin{aligned}(v_1 z^* v U_\sigma)^{2\ell_1+\ell_2} F_{-\ell+i} x &= (v_1 z^* v U_\sigma)^{2\ell_1+\ell_2} (\otimes_{-n-k-2\ell_1-2\ell_2+i}^{n+k+i} p) x F_{-\ell_1-\ell_2+i} \\ &= (\otimes_{-n-k-\ell_2+i}^{n+k+2\ell_1+\ell_2+i} p) x (v_1 z^* v U_\sigma)^{2\ell_1+\ell_2} F_{-\ell_1-\ell_2+i} \\ &= x (v_1 z^* v U_\sigma)^{2\ell_1+\ell_2} F_{-\ell+i}.\end{aligned}$$

Thus we obtain that $F \in D_1'$. By the same computation for $\mathrm{Ad}\, v_1 z^* v U_\sigma(F)$, we also obtain that $\mathrm{Ad}\, v_1 z^* v U_\sigma(F) \in D_1'$. Hence, since $v_1 z^* = z^* v_1$ and $z \in D_1$, it follows that $\mathrm{Ad}\, v_1 v U_\sigma(F) = \mathrm{Ad}\, v_1 z^* v U_\sigma(F) \in D_1'$. Then we get a unitary $V \in (\otimes_{\mathbf{Z}}A_4 \times_\sigma \mathbf{Z}) \cap D_1'$ such that $\mathrm{Ad}\, V v_1 v U_\sigma(F) = F$ and $V - 1 \sim 1/\sqrt{\ell_2}$. Taking $V v_1 v$ for V and D_1 for D and noting that $v_1 - 1 \sim 0$ and $v - 1 \sim 1/k$, we conclude the proof. $\square$

Lemma 2.6. *Let A be a unital approximately divisible* AF *C^*-algebra. For any $\varepsilon > 0$ there exists a 2 by 2 matrix C^*-subalgebra C of A such that $[1_C] > (1-\varepsilon)[1]$ in $\mathbf{R} \otimes K_0(A)$.*

Proof. Since A is approximately divisible, the range of $K_0(A)$ is dense in $\mathrm{Aff}(T(A))$ for the natural map $K_0(A) \to \mathrm{Aff}(T(A))$. Hence there is a positive $g \in K_0(A)$ such that $2^{-1}(1-\varepsilon)[1] < g < 2^{-1}[1]$ in $\mathbf{R} \otimes K_0(A)$. Then we find mutually orthogonal projections e_1 and e_2 in A such that $[e_1] = [e_2] = g$, and choose a $v \in A$ such that $v^*v = e_1$ and $vv^* = e_2$. The C^*-subalgebra generated by v gives the desired C. $\square$

Lemma 2.7. *Let A be a unital simple* AF *C^*-algebra with a unique tracial state and let (A_n) be as usual. For any $n \in \mathbf{N}$ and $\varepsilon > 0$ there exist a projection $e \in \otimes_{\mathbf{Z}}(A \cap A_1')$, a 2 by 2 matrix C^*-subalgebra C of $\otimes_{\mathbf{Z}}A \cap A_1'$, and a projection F and a unitary V in $(\otimes_{\mathbf{Z}}A) \times_\sigma \mathbf{Z}$ such that $e, F \in C'$, $F \in (\otimes_{-n}^n A_1)'$, $1_C \geq F \geq e$, $\|V - 1\| < \varepsilon$, $\mathrm{Ad}\, VU_\sigma | CF = \mathrm{id}$, and $[e] > (1-\varepsilon)[1]$ in $\mathbf{R} \otimes K_0(\otimes_{\mathbf{Z}}A \cap A_1')$.*

Proof. This will be proven just as Lemma 2.5 is.

As in the proof of Lemma 2.5, we obtain a projection $p \in A_2 \cap A_1'$ and a full matrix C^*-subalgebra D of pA_2p such that $D \supset A_1 p$ and then define $D_1 = \mathrm{Ad}\, u(\otimes_{-n-k}^n D \otimes \otimes_{n+1}^{n+k} p) (\subset \otimes_{\mathbf{Z}}A_3)$ for some unitary $u (\in \otimes_{\mathbf{Z}}A_3 \cap \{p\}')$ and a large $k \in \mathbf{N}$, and unitaries $v (\in \otimes_{\mathbf{Z}}A_3 \cap \{p\}')$ and $z (\in D_1 + 1)$; in particular $\|v - 1\| \sim 1/k$, $D_1 \supset (\otimes_{-n}^n A_1) 1_D$, and $\mathrm{Ad}\, z^* v U_\sigma(p_{-n-k-1} x) = p_{n+k+1} x$ for $x \in D_1$.

Since $A \cap A_3'$ is approximately divisible, the previous lemma gives a projection q and a 2 by 2 matrix C^*-subalgebra C in $A \cap A_3'$ such that $q = 1_C$ and $[q] > (1-\varepsilon)[1]$ in $\mathbf{R} \otimes K_0(A \cap A_3')$ for a sufficiently small $\varepsilon > 0$. Here we may replace A by A_4. Then just as in the previous paragraph, we define $C_1 = \mathrm{Ad}\, u'(\otimes_{-n-k}^n C \otimes \otimes_{n+1}^{n+k} q)$ for the same k as above and for some unitary $u' (\in \otimes_{\mathbf{Z}}A_5 \cap A_3')$, and also define unitaries $v'(\in \otimes_{\mathbf{Z}} A_5 \cap A_3')$ and $z'(\in C_1+1)$; in particular $||v'-1||\sim 1/k$, v' commutes with q at any point of $\mathbf{Z}$, and $\mathrm{Ad}\, z'^* v' U_\sigma(q_{-n-k-1}x) = q_{n+k+1}x$ for $x\in C_1$.

Let $\ell_1,\ell_2\in\mathbf{N}$ be such that $\ell_1\gg\ell_2\gg 1$ and let $\ell=\ell_1+\ell_2$. By the Rohlin property σ on $\otimes_{\mathbf{Z}} A_6\cap A_5'$, we obtain an orthogonal family $(f_i)_{i=-\ell}^{\ell}$ of projections and a unitary $v_1\in\otimes_{\mathbf{Z}} A_6\cap A_5'$ such that $v_1\sim 1$, $\mathrm{Ad}\, v_1\sigma(f_i)=f_{i+1}$, and $[f_i]>(2\ell+2)^{-1}[1]$ in $\mathbf{R}\otimes K_0(\otimes_{\mathbf{Z}} A_6\cap A_5')$. We define, for $i=-\ell,-\ell+1,\dots,\ell$,

$$F_i = f_i(\otimes_{-n-k-\ell+i}^{n+k+\ell+i} pq).$$

Here (F_i) is an orthogonal family of projections such that $\mathrm{Ad}\, v_1 z'^* v' z^* v U_\sigma(F_i) = F_{i+1}$. We assume that $[F_i]>(2\ell+2)^{-1}[1]$ in $\mathbf{R}\otimes K_0(\otimes_{\mathbf{Z}} A_6\cap A_1')$ by assuming that $[pq]$ is sufficiently close to $[1]$ in $\mathbf{R}\otimes K_0(A_6\cap A_1')$. We define an almost $\mathrm{Ad}\, v_1 z'^* v' z^* v U_\sigma$-invariant projection $F\in\otimes_{\mathbf{Z}} A_6\times_\sigma\mathbf{Z}$ in terms of (F_i) and $(v_1 z'^* v' z^* v U_\sigma)^{2\ell_1+\ell_2}$ just as in the proof of Lemma 2.5. Since $F, \mathrm{Ad}\, v_1 z'^* v' z^* v U_\sigma(F)\in C^*(C_1,D_1)'$, we find a unitary $V_1\in\otimes_{\mathbf{Z}} A_6\times_\sigma\mathbf{Z}$ such that $V_1\in C^*(C_1,D_1)'$, $V_1\sim 1$, and $\mathrm{Ad}\, V_1 v_1 z'^* v' z^* v U_\sigma(F)=F$. Note that $\mathrm{Ad}\, V_1 v_1 z'^* v' z^* v U_\sigma | C^*(C_1,D_1)F=\mathrm{id}$. Since $\mathrm{Ad}\, z'|C_1$ has period $2n+k+1$, the spectral gap of z' is no greater than $2\pi/(2n+k+1)$. Hence there is a 2 by 2 matrix C^*-subalgebra C_2 of C_1 such that $||(\mathrm{Ad}\, z'-\mathrm{id})|C_2||$ is at most of the order of $1/(2n+k+1)$. Since $\mathrm{id}|C_2=\mathrm{Ad}\, V_1 v_1 z'^* v' z^* v U_\sigma|C_2=\mathrm{Ad}\, V_1 z'^* v' U_\sigma|C_2\simeq\mathrm{Ad}\, U_\sigma|C_2$, C_2 has the desired properties for C. The other properties are also satisfied just as in the proof of Lemma 2.5. $\square$

Lemma 2.8. *Let A be a unital simple* AF *C^*-algebra with a unique tracial state and let (A_n) be as usual. For any $n\in\mathbf{N}$ and $\varepsilon>0$ there exists a C^*-subalgebra C of $(\otimes_{\mathbf{Z}} A)\times_\sigma\mathbf{Z}$ and a unitary $V\in(\otimes_{\mathbf{Z}} A)\times_\sigma\mathbf{Z}$ such that $C\ni 1$, $C\subset(\otimes_{-n}^n A_1)'$, $C\cong M_2\oplus M_3$, $||V-1||<\varepsilon$, and* $\mathrm{Ad}\, VU_\sigma|C=\mathrm{id}$.

Proof. We use the previous lemma for a sufficiently small $\varepsilon>0$, where in the conclusion we may replace A by A_k for a sufficiently large k. By changing indices we may assume that $k=2$.

Let $\ell\in\mathbf{N}$ be such that $\ell\gg 1$ and $\ell<(3\varepsilon)^{-1}$. By using the Rohlin property for σ on $\otimes_{\mathbf{Z}} A\cap A_2'$, we find an orthogonal family $(f_i)_{i=0}^{\ell-1}$ of projections and a unitary v in $\otimes_{\mathbf{Z}} A\cap A_2'$ such that $v\cong 1$, $\mathrm{Ad}\, vU_\sigma(f_i)=f_{i+1}$, and $[f_i]>(1+\ell)^{-1}[1]$ in $\mathbf{R}\otimes K_0(\otimes_{\mathbf{Z}} A\cap A_2')$. Let (e_{ij}) be a set of matrix units in C. We may assume that $[1-e]\leqslant[e_{11}ef_1]$ in $K_0(\otimes_{\mathbf{Z}} A\cap A_1')$ and let $b\in\otimes_{\mathbf{Z}} A\cap A_1'$ be a partial isometry such that $b^*b=1-e$ and $bb^*\leqslant e_{11}ef_1$. We define

$$y=\frac{1}{\sqrt{\ell}}\sum_{i=0}^{\ell-1}(vU_\sigma)^i b(vU_\sigma)^{-i}(1-F).$$

Then y belongs to $(\otimes_{\mathbf{Z}} A \cap A_1')(1-F) \subset (\otimes_{-n}^{n} A_1)'$, is close to a partial isometry

$$y_1 = \frac{1}{\sqrt{\ell}} \sum_{i=0}^{\ell-1} (vU_\sigma)^i b (VU_\sigma)^{-i}(1-F)$$

with $y_1^* y_1 = 1-F$, and is close to

$$y_2 = \frac{1}{\sqrt{\ell}} \sum_{i=0}^{\ell-1} (VU_\sigma)^i b (vU_\sigma)^{-i}(1-F)$$

with $y_2 = Fe_{11} y_2$. Note also that

$$||\mathrm{Ad}\, vU_\sigma(y) - y|| \leqslant 2/\sqrt{\ell}.$$

Let v_1 be the partial isometry obtained by the polar decomposition of $Fe_{11} y \in (\otimes_{-n}^{n} A_1)'$. Then it follows that $v_1^* v_1 = 1-F$, $v_1 v_1^* \leqslant Fe_{11}$, and $[U_\sigma, v_1] \sim 0$. Then the C^*-subalgebra C_1 generated by CF and v_1 satisfies that $C_1 \subset (\otimes_{-n}^{n} A_1)'$, $C_1 \ni 1$, $C_1 \cong M_2 \oplus M_3$, and $\mathrm{Ad}\, U_\sigma | C_1 \cong \mathrm{id}$. This concludes the proof. $\square$

Proof of Theorem 2.3. Since σ is asymptotically abelian (or has the Rohlin property), the crossed product $(\otimes_{\mathbf{Z}} A) \times_\sigma \mathbf{Z}$ has a unique tracial state (see [1] for more results).

By Lemma 2.8 we know that the crossed product $(\otimes_{\mathbf{Z}} A) \times_\sigma \mathbf{Z}$ is approximately divisible. Since the crossed product has a unique tracial state, it has real rank zero [2]. In the situation of Lemma 2.5 let $z \in D$ be a unitary such that $\mathrm{Ad}\, VU_\sigma | DF = \mathrm{Ad}\, z | DF$. Then $z^* VU_\sigma F$ is a unitary in $FD'F$; so the C^*-subalgebra generated by DF and $z^* VU_\sigma F$ is isomorphic to $D \otimes C(\mathbf{T})$ or a quotient of it and contains $(\otimes_{-n}^{n} A_1)F$ exactly and $FU_\sigma F \cong U_\sigma F$ almost (since $V \cong 1$). (One can show that the K_1 class of $z^* VU_\sigma F + 1 - F$ is non-zero if ε is small and so that its spectrum is full. But, since this fact is not required, we will refrain from proving it.) Thus $(\otimes_{\mathbf{Z}} A) \times_\sigma \mathbf{Z}$ is *tracially* AT.

Since $W = z^* VU_\sigma F$ is a unitary in the C^*-algebra $F((\otimes_{\mathbf{Z}} A) \times_\sigma \mathbf{Z})F \cap FD'F$ of real rank zero, one can approximate W by a unitary $W_1 + W_2$ such that $G = W_1^* W_1 = W_1 W_1^*$ is a projection close to F, i.e., $\tau(G) \simeq \tau(F)$ with τ the tracial state, and $\mathrm{Sp}(W_1)$ is finite. Since $D \supset (\otimes_{-n}^{n} A_1) 1_D$ and $F \in (\otimes_{-n}^{n} A_1)'$, we have that $G \in (\otimes_{-n}^{n} A_1)'$. Since $[W, G] \simeq 0$, we have that $GU_\sigma = zGz^* U_\sigma \simeq zGz^* VU_\sigma = zGW \simeq zWG \simeq U_\sigma G$. Let D_1 be the C^*-subalgebra generated by GDG and W_1. Then D_1 is finite-dimensional and its unit G commutes with $\otimes_{-n}^{n} A_1$ and almost commutes with U_σ. It follows that

$$G(\otimes_{-n}^{n} A_1) \subset GDG \subset D_1$$

and

$$GU_\sigma G \simeq GzW_1 \in D_1.$$

This concludes the proof that $(\otimes_{\mathbf{Z}} A) \times_\sigma \mathbf{Z}$ is a tracially AF C^*-algebra.

3. Non-unital tensor products

If we are given a C^*-algebra $A(i)$ and a non-zero projection $e_i \in A(i)$ for each $i \in \mathbf{Z}$, we define a C^*-algebra $\otimes_{i\in\mathbf{Z}}(A(i), e_i)$ as the inductive limit of the following inductive system $(A_\Lambda, \varphi_{\Lambda_1,\Lambda_2})$: For each finite set Λ of $\mathbf{Z}$ let A_Λ be the (minimal) tensor product of $\{A(i),\ i\in\Lambda\}$ and for $\Lambda_1 \subset \Lambda_2$ define an embedding $\varphi_{\Lambda_1,\Lambda_2}$ of A_{Λ_1} into A_{Λ_2} by $x \mapsto x \otimes (\otimes_{i\in\Lambda_2\setminus\Lambda_1} e_i)$. If $A(i)$ is unital and $e_i = 1$ for every i, then $\otimes_{i\in\mathbf{Z}}(A(i), e_i)$ is just the usual tensor product $\otimes_{i\in\mathbf{Z}} A(i)$.

Suppose that all $A(i)$'s are unital but $e_i \neq 1$ for infinitely many i. Then $\otimes_{i\in\mathbf{Z}}(A(i), e_i)$ is a non-unital C^*-algebra and $\otimes_{i\in\mathbf{Z}} A(i)$ can be naturally regarded as a multiplier algebra of $\otimes_{i\in\mathbf{Z}}(A(i), e_i)$. Suppose that we are given another projection $e_i' \in A(i)$ for each i and if e_i' is unitarily equivalent to e_i, then $\otimes_{i\in\mathbf{Z}}(A(i), e_i')$ is isomorphic to $\otimes_{i\in\mathbf{Z}}(A(i), e_i)$. If $e_i' = e_i$ except for a finite number of i, then $\otimes_{i\in\mathbf{Z}}(A(i), e_i')$ and $\otimes_{i\in\mathbf{Z}}(A(i), e_i)$ are identical.

Suppose that we are given a C^*-algebra A and that $A(i) = A$. Let e_- and e_+ be non-zero projections in A and let $e_i = e_+$ for $i \geqslant 0$ and $e_i = e_-$ for $i < 0$. In this case since $e_{i+1} = e_i$ except for $i = -1$, we can define a shift automorphism σ of $\otimes_{\mathbf{Z}}(A, e_-, e_+) \equiv \otimes_{i\in\mathbf{Z}}(A(i), e_i)$ by $x \in A(i) \mapsto x \in A(i+1)$.

Suppose that we are given another pair e_-', e_+' of projections in A such that $e_\pm'$ is unitarily equivalent to $e_\pm$ (in $A + \mathrm{C}1$ if A is non-unital). Then denoting by σ' the shift automorphism of $\otimes_{\mathbf{Z}}(A, e_-', e_+')$, it follows that $(\otimes_{\mathbf{Z}}(A, e_-, e_+), \sigma)$ and $(\otimes_{\mathbf{Z}}(A, e_-', e_+'), \sigma')$ are outer conjugate. To see this let $U_\pm$ be unitaries in A such that $\mathrm{Ad}\, U_\pm(e_\pm) = e_\pm'$. With $U_i = U_+$ for $i \geqslant 0$ and $U_i = U_-$ for $i < 0$, define a map φ of $\otimes_{\mathbf{Z}}(A, e_-, e_+)$ into $\otimes_{\mathbf{Z}}(A, e_-', e_+')$ by $x \in A(i) \mapsto \mathrm{Ad}\, U_i(x) \in A(i)$. This is indeed an isomorphism. Since $\varphi^{-1}\sigma'\varphi\sigma^{-1} = \otimes_i \mathrm{Ad}\, U_i^* U_{i-1}$, and $U_i^* U_{i-1} = 1$ except for $i = 0$, $\varphi^{-1}\sigma'\varphi\sigma^{-1}$ is inner, proving the assertion. Hence in particular $\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z}$ is isomorphic to $\otimes_{\mathbf{Z}}(A, e_-', e_+') \times_\sigma \mathbf{Z}$.

Suppose that A is a non-unital AF C^*-algebra and e_-, e_+ are projections in A. By changing e_-, e_+ by equivalent projections if necessary, we may assume that there is an approximate unit (e_n) consisting of projections such that $e_1 \geqslant e_-, e_+$. Then

$$(\otimes_{\mathbf{Z}}(e_n A e_n, e_-, e_+))_n$$

is an increasing sequence of σ-invariant hereditary C^*-subalgebras of $\otimes_{\mathbf{Z}}(A, e_-, e_+)$ with dense union. Thus it follows that $(\otimes_{\mathbf{Z}}(e_n A e_n, e_-, e_+) \times_\sigma \mathbf{Z})$ is also an increasing sequence of hereditary C^*-subalgebras of $\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z}$ with dense union. Thus when we consider the crossed products, there will be no loss of generality by assuming that A is unital.

Proposition 3.1. *If A is an* AF *C^*-algebra and $[e_-] \neq [e_+]$, then $K_1(\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z})$ is zero.*

Proof. We have to show that the kernel of $\mathrm{id} - \sigma_*$ on $K_0(\otimes_{\mathbf{Z}}(A, e_-, e_+))$ is $\{0\}$. Suppose that $g = \sigma_*(g)$ for some $g \in K_0(\otimes_{\mathbf{Z}}(A, e_-, e_+))$. There is an $n \in \mathbf{N}$ such that

$g \in K_0(\otimes_{-n}^{n}(A, e_-, e_+)) \subset K_0(\otimes_{\mathbf{Z}}(A, e_-, e_+))$. Since $g = \sigma_*^{2n+1}(g)$ and the $n+1$st factor of g (resp. $\sigma_*^{2n+1}(g)$) is $[e_+]$ (resp. $[e_-]$), if $[e_-]$ and $[e_+]$ are rationally independent, it follows that $g = 0$. If $m[e_+] = \ell[e_-]$ with $m \neq \ell$, then deleting the common factor $m[e_+] = \ell[e_-]$ at $n+1 \in \mathbf{Z}$ from mg and $\ell\sigma_*^{2n+1}(g)$, we obtain g and $\sigma_*^{2n}(g)$ respectively. Thus $mg = \ell\sigma_*^{2n+1}(g) = \ell g$. Hence it follows that $g = 0$. $\square$

Proposition 3.2. *If A is an* AF *C^*-algebra, $[e_-] \neq [e_+]$, and if $p \mid [e_-] - [e_+]$ then $p \mid [e_-]$ and $p \mid [e_+]$ for all prime numbers p, then $K_0(\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z})$ is torsion-free.*

Proof. We have to show that $K_0(\otimes_{\mathbf{Z}}(A, e_-, e_+))/\mathrm{Range}(\mathrm{id} - \sigma_*)$ is torsion-free.

Let F be the subfield of $\boldsymbol{Q}$ generated by

$$\{n \in \mathbf{N};\ n \text{ divides } [e_-] \text{ or } [e_+]\}.$$

Then $K_0(\otimes_{\mathbf{Z}}(A, e_-, e_+))$ is a module over F and σ_* is a module homomorphism.

Suppose that $K_0(\otimes_{\mathbf{Z}}(A, e_-, e_+))/\mathrm{Range}(\mathrm{id} - \sigma_*)$ has torsion and let h be an element not in the range of $\mathrm{id} - \sigma_*$ but $nh = g - \sigma_*(g)$ for some $n > 1$ and $g \in K_0(\otimes_{\mathbf{Z}}(A, e_-, e_+))$. We suppose that n is the smallest positive integer with this property. In particular no prime factors of n appear in F. Since $g - \sigma_*^k(g)$ is divisible by n for any $k \in \mathbf{N}$, we are led to the following situation: If $G = K_0(A)^{\otimes m}$ for some $m \in \mathbf{N}$, there is a $g \in G$ such that $\xi_- \otimes g - g \otimes \xi_+$ is divisible by n, where $\xi_\pm = [e_\pm]^{\otimes m}$. We may then replace $K_0(A)$ by $\mathbf{Z}^\ell$ for some $\ell \in \mathbf{N}$ and so G by $\mathbf{Z}^{m\ell}$. Our standing assumption says that no prime factors of n divide either ξ_- or ξ_+.

Let p be a prime factor of n and let s be the maximum integer such that $p^s \mid n$. If there is a component g_i of g such that p^s does not divide g_i, then by the same argument used in the proof of A1, we have that $g = p^{s-t}g' + d\xi_+$, where $s - t > 0$. If $p \mid d$ then $p \mid g$, which contradicts the choice of n. Thus we are led to the situation that $\xi_- \otimes \xi_+ - \xi_+ \otimes \xi_+$ is divisible by p, which means $\xi_- - \xi_+$ is divisible by p (since ξ_+ is not). Since $[e_-] - [e_+]$ is not divisible by p by the assumption, it follows that $[e_-]^{\otimes m} - [e_+]^{\otimes m}$ is not. Hence we have reached a contradiction. $\square$

Theorem 3.3. *Let A be a unital simple* AF *C^*-algebra with a unique tracial state τ and let e_-, e_+ be non-zero projections in A such that $\tau(e_-) = \tau(e_+) < 1$. Then $\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z}$ is a simple tracially* AF *C^*-algebra, which admits a densely defined lower semi-continuous trace, unique up to constant multiples.*

There is a (unique up to constant multiples, densely defined lower semi-continuous) trace on $\otimes_{\mathbf{Z}}(A, e_-, e_+)$ which is left invariant under σ. Hence the part on trace is obvious in the above assertion.

Let (A_n) be an increasing sequence of finite-dimensional C^*-subalgebras of A with $A = \overline{\cup_n A_n}$ and $A_1 \ni 1$. Let $e_-, e_+ \in A_1$ be projections such that $0 < \tau(e_-) = \tau(e_+) < 1$. For $m \leqslant n$ we denote the identity of $\otimes_m^n A$ in $\otimes_{\mathbf{Z}}(A, e_-, e_+)$ by $1_{(m,n)}$. We shall show that for any $n \in \mathbf{N}$ and $\varepsilon > 0$ there is a C^* subalgebra C of $\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z}$

such that

$$1_C = 1_{(-n,n)},$$

$$C \cong M_2 \oplus M_3,$$

$$C \subset (\otimes_{-n}^{n}(A_1, e_-, e_+))',$$

$$\|[x, U_\sigma 1_{(-n,n-1)}]\| < \varepsilon\|x\|, \quad x \in C.$$

This implies that for any finite subset $\mathscr{F}$ of $\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z}$ and any $\varepsilon > 0$ there is a C^* subalgebra C such that $\|(1 - 1_C)y\| < \varepsilon$ for $y \in \mathscr{F}$, $\|[x, y]\| < \varepsilon\|x\|$ for $x \in C$ and $y \in \mathscr{F}$, and $C \cong M_2 \oplus M_3$. Hence it follows that for any projection $p \in \otimes_{\mathbf{Z}} (A, e_-, e_+) \times_\sigma \mathbf{Z}$, $p(\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z})p$ is approximately divisible and thus has real rank zero. Then one can conclude that $\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z}$ has real rank zero (or $\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z} + \mathbf{C}1$ has real rank zero [9]).

In Lemma 3.5 we shall show that $\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z}$ is *tracially* AT, i.e., for any $n \in \mathbf{N}$ and $\varepsilon > 0$ there is a C^*-subalgebra D of $\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z}$ such that D is isomorphic to (a quotient of) the tensor product of a full matrix algebra and $C(\mathbf{T})$, and

$$1_D \in (\otimes_{-n}^{n} A_1)',$$

$$1_D \leqslant 1_{(-n,n)},$$

$$[1_{(-n,n)} - 1_D] \leqslant \varepsilon[1_{(-n,n)}],$$

$$D \supset (\otimes_{-n}^{n} A_1)1_D,$$

$$\|[1_D, U_\sigma 1_{(-n,n-1)}]\| < \varepsilon,$$

$$\mathrm{dist}(D, 1_D U_\sigma 1_{(-n,n-1)} 1_D) < \varepsilon.$$

Since $(1_{(-n,n)})$ forms an approximate identity for $\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z}$, this implies that for any projection $p \in \otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z}$, the hereditary C^*-subalgebra cut down by p is *tracially* AT. Using the fact that $\otimes_{\mathbf{Z}}(A, e_-, e_+) \times_\sigma \mathbf{Z}$ has real rank zero, which will be shown later, we conclude that it is tracially AF, in the same way as in the unital case. Note also that if $[e_-] \neq [e_+]$, then $\otimes_{\mathbf{Z}}(A_k, e_-, e_+) \times_\sigma \mathbf{Z}$ has purely infinite quotients for any k.

First we shall give an analogue of 2.4.

Lemma 3.4. *Let A be a unital simple* AF *C^*-algebra with a unique tracial state and let (A_n) and $e_-, e_+ \in A_1$ as above. Then for any $\varepsilon > 0$ there exist a $k \in \mathbf{N}$, a projection*

$p \in A_k \cap A_1'$, and a full matrix C^*-subalagebra D of pA_kp such that $D \ni p$, $D \supset A_1 p$, $[pe_+] = [pe_-]$ in $K_0(A_k)$, and $[p] > (1-\varepsilon)[1]$ in $\mathbf{R} \otimes K_0(A_k \cap A_1')$.

Proof. If $[e_-] = [e_+]$ then we may assume that $[e_-] = [e_+]$ in $K_0(A_1)$ and then this is just Lemma 2.4. Assuming that $[e_-] \neq [e_+]$ we will extend Lemma 2.4, adopting the same notation as in its proof.

When we choose $c_j, d_j \in \mathbf{N}$ for $j = 1, 2, \dots, k_1$ there (such that c_j/d_j approximates $\mu_j/\xi_1(j)$), we have to impose an extra condition corresponding to $[pe_+] = [pe_-]$; i.e., we have to find $c_j, d_j \in \mathbf{N}$ such that $c_j \gg 1$ and

$$\frac{c_j}{d_j} < \frac{\mu_j}{\xi_1(j)} < \frac{c_j + 1}{d_j},$$

$$\sum_{j=1}^{k_1} c_j \dim(e_- p_{1j}) = \sum_{i=1}^{k_1} c_j \dim(e_+ p_{1j}),$$

where $\dim(e_\pm p_{1j})$ is the dimension of $e_\pm p_{1j}$ in $A_1 p_{1j}$. Since

$$\sum_j \frac{\mu_j}{\xi_1(j)} \dim(e_- p_{1j}) = \sum_j \frac{\mu_j}{\xi_1(j)} \dim(e_+ p_{1j}),$$

which follows from $\tau(e_-) = \tau(e_+)$, both

$$\{j;\ \dim(e_- p_{1j}) > \dim(e_+ p_{1j})\} \quad \text{and} \quad \{j;\ \dim(e_- p_{1j}) < \dim(e_+ p_{1j})\}$$

are non-empty. Hence we can find $c_j \in \mathbf{N}$ such that $c_j \gg 1$ and

$$\sum_{i=1}^{k_1} c_j(\dim(e_- p_{1j}) - \dim(e_+ p_{1j})) = 0.$$

By assuming that $\xi_1(j)/\mu_j > 1$, we can then find $d_j \in \mathbf{N}$ satisfying

$$\frac{\xi_1(j)}{\mu_j} c_j < d_j < \frac{\xi_1(j)}{\mu_j}(c_j + 1).$$

Having defined c_j, d_j we can proceed as in the proof of 2.4. □

Lemma 3.5. *Let A be a unital simple* AF *C^*-algebra with a unique tracial state τ and let (A_n) and $e_-, e_+ \in A_1$ as above. For any $n \in \mathbf{N}$ and $\varepsilon > 0$ there exist a $k \in \mathbf{N}$, a projection $e \in \otimes_{\mathbf{Z}}(A_k \cap A_1')$, a projection $g \in \otimes_{\mathbf{Z}}(A_k, e_-, e_+)$, a full matrix C^*-subalgebra D of $\otimes_{\mathbf{Z}} A_k$, a projection $F \in (\otimes_{\mathbf{Z}} A_k) \times_\sigma \mathbf{Z}$, and a unitary V in the multiplier algebra of $\otimes_{\mathbf{Z}}(A_k, e_-, e_+) \times_\sigma \mathbf{Z}$ such that*

$$1_D \in \otimes_{\mathbf{Z}} A_k \cap A_1',$$

$$F, e, g \in D',$$

$$[F, g] = 0,$$

$$[e, g] = 0,$$

$$1_D \geqslant F \geqslant e,$$

$$g \geqslant 1_{(-n,n)},$$

$$D \supset (\otimes_{-n}^{n} A_1) 1_D,$$

$$Dg \supset (\otimes_{-n}^{n} (A_1, e_-, e_+)) 1_D,$$

$$\|V - 1\| < \varepsilon,$$

$$\mathrm{Ad}\, V U_\sigma(Fg) = Fg,$$

$$\mathrm{Ad}\, V U_\sigma(DF) = DF,$$

$$[e] > (1-\varepsilon)[1] \ \textit{in}\ \mathbf{R} \otimes K_0(\otimes_{\mathbf{Z}} A_k \cap A_1').$$

Proof. This will be proven just as Lemma 2.5 is. We use the same notation as in its proof.

We select a projection $p \in A_2 \cap A_1'$ and a full matrix C^*-subalgebra $D \subset pA_2p$ with $D \supset A_1 p$ as in Lemma 3.4.

Let $k \in \mathbf{N}$ with $k \geqslant 1$. We work in the multiplier algebra of $\otimes_{\mathbf{Z}}(A, e_-, e_+)$. Let w be a unitary in $\otimes_{-n-k}^{n+1} A_2$ such that $w(1 - \otimes_{-n-k}^{n+1} p) = 1 - \otimes_{-n-k}^{n+1} p$, $w(\otimes_{-n-k}^{n+1} p) \in D \otimes (\otimes_{-n-k+1}^{n} p) \otimes D$, and $\mathrm{Ad}\, w | \otimes_{-n-k}^{n+1} p$ switches D at $-n-k$ and D at $n+1$ in such a way that it switches $(pe_-)_{-n-k}$ and $(pe_+)_{n+1}$. This is possible because D is a full matrix algebra and $\dim(pe_-) = \dim(pe_+)$ in D. Hence in particular $\mathrm{Ad}\, w$ leaves $(pe_-)_{-n-k}(pe_+)_{n+1}$ invariant. By using a set of Rohlin towers for σ on $\otimes_{\mathbf{Z}} A_3 \cap A_2'$ (which is in the multiplier algebra) we will choose a unitary u in the multiplier algebra just as in the proof of 2.5. For example u belongs to the C^*-subalgebra generated by $\otimes_{-n-k}^{-n-2} A_2 \cap \{p\}' \otimes (\otimes_{-n-1}^{n} \{1, p\}'') \otimes_{n+1}^{n+k-1} A_2 \cap \{p\}'$ and $\otimes_{\mathbf{Z}} A_3 \cap A_2'$ and satisfies that $\|w - u^* \sigma(u)\| \simeq 1/k$. This time we can impose an extra condition that u commutes with $(pe_-)_{-n-k+i}(pe_+)_{n+1+i}$ for $i = 0, 1, \ldots, k-2$.

Let

$$D_1 = \mathrm{Ad}\, u(\otimes_{-n-k}^{n} D \otimes (\otimes_{n+1}^{n+k} p)).$$

Note that D_1 is in the multiplier algebra, $1_{D_1} = \otimes_{-n-k}^{n+k} p$, and $D_1 \supset (\otimes_{-n}^{n} A_1) 1_{D_1}$. Let $g = \mathrm{Ad}\, u(1_{(-n-k,n)})$, which is a projection in the commutant of D_1 and satisfies that $g \geqslant \mathrm{Ad}\, u(1_{(-n,n)}) = 1_{(-n,n)}$.

Setting $v = uw\sigma(u^*)$, we have a unitary $z \in D_1 + 1$ such that $\mathrm{Ad}\, v\sigma(p_{-n-k-1}x) = \mathrm{Ad}\, z(x)p_{n+k+1}$ for $x \in D_1$. We should note that if $x \in D_1 g$, then $\mathrm{Ad}\, v\sigma((pe_-)_{-n-k-1}x) = \mathrm{Ad}\, z(x)(pe_+)_{n+k+1}$.

For $\ell = \ell_1 + \ell_2$ with $\ell_1 \gg \ell_2 \gg 1$ and $\ell = \ell_1 + \ell_2$, we have an orthogonal family $(f_i)_{i=-\ell}^{\ell}$ of projections in $\otimes_{\mathbf{Z}} A_4 \cap A_3'$ and a unitary $v_1 \in \otimes_{\mathbf{Z}} A_4 \cap A_3'$ such that $v_1 \simeq 1$, $\mathrm{Ad}\, v_1\sigma(f_i) = f_{i+1}$, and $[f_i] > 1/(2\ell + 2)[1]$. We define $F_i = f_i(\otimes_{-n-k-\ell+i}^{n+k+\ell+i} p)$ just as in 2.5. Then we should note that g commutes with F_i and that $\mathrm{Ad}\, v_1 z^* v U_\sigma(F_i g) = F_{i+1} g$ as well as $\mathrm{Ad}\, v_1 z^* v U_\sigma(F_i) = F_{i+1}$. We define an almost $\mathrm{Ad}\, v_1 z^* v U_\sigma$-invariant projection $F \in (\otimes_{\mathbf{Z}} A_4) \times_\sigma \mathbf{Z}$ in terms of (F_i) and $(v_1 z^* v U_\sigma)^{2\ell_1+\ell_2}$ just as before. In particular we have that $F \geqslant \sum_{i=-\ell_1}^{\ell_1} F_i \equiv e$ and that $F \leqslant \otimes_{-n-k-1}^{n+k+1} p \leqslant 1_{D_1}$. We should notice that g commutes with F, e and that Fg can be defined just in the same way as F in terms of $(F_i g)$ and $(v_1 z^* v U_\sigma)^{2\ell_1+\ell_2}$. By the same computation as before we can see that $F, Fg, \mathrm{Ad}\, v_1 z^* v U_\sigma(F), \mathrm{Ad}\, v_1 z^* v U_\sigma(Fg)$ commutes with D_1. Since $\mathrm{Ad}\, v_1 z^* v U_\sigma(F) = \mathrm{Ad}\, v_1 v U_\sigma(F)$ and $\mathrm{Ad}\, v_1 z^* v U_\sigma(Fg) = \mathrm{Ad}\, v_1 v U_\sigma(Fg)$, we get a unitary V in the commutant of D_1 in the multiplier algebra with V such that $\mathrm{Ad}\, V v_1 v U_\sigma(F) = F$ and $\mathrm{Ad}\, V v_1 v U_\sigma(Fg) = Fg$. Taking $V v_1 v$ for V and D_1 for D, we can check all the other properties.

Finally we indicate how the assertion made after Theorem 3.3 follows. Let z be a unitary in $D(\equiv D_1)$ such that $\mathrm{Ad}\, VU_\sigma(xF) = \mathrm{Ad}\, z(x)F$ for $x \in D$. Then, since $z^* VU_\sigma$ is in the commutant of DFg and $1_{(-n,n)} 1_D$ is a projection in Dg, the C^*-subalgebra generated by $1_{(-n,n)} D 1_{(-n,n)} F$ and $z^* VU_\sigma F 1_{(-n,n)}$ is isomorphic to $1_{(-n,n)} D 1_{(-n,n)} \otimes C(\mathbf{T})$ (or a quotient of it) and its identity $1_{(-n,n)} F$ is close to $1_{(-n,n)}$ in the sense that $[1_{(-n,n)} - 1_{(-n,n)} F] \leqslant [1_{(-n,n)}(1-e)] < \varepsilon [1_{(-n,n)}]$ in $\mathbf{R} \otimes K_0(\otimes_{\mathbf{Z}} (A_k, e_-, e_+))$. We should also note that $F 1_{(-n,n)}$ commutes with $VU_\sigma 1_{(-n,n-1)} (\simeq U_\sigma 1_{(-n,n-1)})$. $\square$

Lemma 3.6. *Let A be a unital simple* AF *C^*-algebra with a unique tracial state and let (A_n) and $e_-, e_+ \in A_1$ be as before. For any $n \in \mathbf{N}$ and $\varepsilon > 0$ there exist a projection $e \in \otimes_{\mathbf{Z}} A \cap A_1'$, a 2 by 2 matrix C^*-subalgebra C of $\otimes_{\mathbf{Z}} A \cap A_1'$, a projection $F \in \otimes_{\mathbf{Z}} A \times_\sigma \mathbf{Z}$, and a unitary V in the multiplier algebra of $\otimes_{\mathbf{Z}} A \times_\sigma \mathbf{Z}$ such that*

$$e, F \in C',$$

$$[F, 1_{(-n,n)}] = 0,$$

$$F \in (\otimes_{-n}^{n} A_1)',$$

$$1_C \geqslant F \geqslant e,$$

$$\|V - 1\| < \varepsilon,$$

$$\mathrm{Ad}\, VU_\sigma | CF = \mathrm{id},$$

$$\mathrm{Ad}\,VU_\sigma(F1_{(-n,n)}) = F1_{(-n,n)},$$

$$[e] > (1-\varepsilon)[1] \text{ in } \mathbf{R}\otimes K_0(\otimes_{\mathbf{Z}}A\cap A_1').$$

Proof. As in the proof of the previous lemma, we obtain a projection $p\in A_2\cap A_1'$ and a full matrix C^*-subalgebra D of pA_2p with $D\supset A_1p$ and then construct D_1, u, v, z, g for a large $k\in\mathbf{N}$ just as there.

Since $A\cap A_3'$ is approximately divisible, Lemma 2.6 gives, for any $\varepsilon>0$, a projection q and a 2 by 2 matrix C^*-subalgebra $C\subset A\cap A_3'$ such that $q=1_C$ and $[q]>(1-\varepsilon)[1]$ in $\mathbf{R}\otimes K_0(A\cap A_3')$. Here we may replace A by A_4. We should note here that $[qpe_-]=[qpe_+]$ in $K_0(A_4)$ since $[pe_-]=[pe_+]$ in $K_0(A_3)$. Then just as in the proof of the previous lemma (or Lemma 2.5), we define $C_1=\mathrm{Ad}\,u'(\otimes_{-n-k}^{n} C\otimes\otimes_{n+1}^{n+k} q)$ for the same k as above and for some unitary $u'(\in\otimes_{\mathbf{Z}}A_5\cap A_3')$, and also defines unitaries $v'(\in\otimes_{\mathbf{Z}}A_5\cap A_3')$ and $z'(\in C_1+1)$; in particular $\|v'-1\|\sim 1/k$, v' commutes with q at any point of $\mathbf{Z}$, and $\mathrm{Ad}\,z'^*v'U_\sigma(q_{-n-k-1}x)=q_{n+k+1}x$ for $x\in C_1$. Then it follows that $\mathrm{Ad}\,z'^*v'z^*vU_\sigma((pq)_{-n-k-1}x)=(pq)_{n+k+1}x$ for $x\in C^*(C_1,D_1)$ and $\mathrm{Ad}\,z'^*v'z^*vU_\sigma((pqe_-)_{-n-k-1}x)=(pqe_+)_{n+k+1}x$ for $x\in C^*(C_1,D_1)g$, where (pqe_-) and (pqe_+) can be replaced by (pq). Note that $1_{(-n,n)}C^*(C_1,D_1)1_{(-n,n)}\subset C^*(C_1,D_1)g$ since $1_{(-n,n)}\in C^*(C_1,D_1)g$.

Let $\ell_1,\ell_2\in\mathbf{N}$ be such that $\ell_1\gg\ell_2\gg 1$ and let $\ell=\ell_1+\ell_2$. By the Rohlin property σ on $\otimes_{\mathbf{Z}}A_6\cap A_5'$, we obtain an orthogonal family $(f_i)_{i=-\ell}^{\ell}$ of projections and a unitary $v_1\in\otimes_{\mathbf{Z}}A_6\cap A_5'$ such that $v_1\simeq 1$, $\mathrm{Ad}\,v_1\sigma(f_i)=f_{i+1}$, and $[f_i]>(2\ell+2)^{-1}[1]$ in $\mathbf{R}\otimes K_0(\otimes_{\mathbf{Z}}A_6\cap A_5')$. We define, for $i=-\ell,-\ell+1,\dots,\ell$,

$$F_i = f_i(\otimes_{-n-k-\ell+i}^{n+k+\ell+i}pq).$$

Note that $\mathrm{Ad}\,v_1z'^*v'z^*vU_\sigma(F_i)=F_{i+1}$ and $\mathrm{Ad}\,v_1z'^*v'z^*vU_\sigma(F_i1_{(-n,n)})=F_{i+1}1_{(-n,n)}$. We can then proceed just as in the proof of the previous lemma (taking $1_{(-n,n)}$ for g this time). See also the proof of 2.7. $\square$

Lemma 3.7. *Let A be a unital simple* AF *C^*-algebra with a unique tracial state and let $(A_n), e_-, e_+\in A_1$ be as usual. For any $n\in\mathbf{N}$ and $\varepsilon>0$, there exists a C^*-subalgebra C of $\otimes_{\mathbf{Z}}(A,e_-,e_+)\times_\sigma\mathbf{Z}$ such that*

$$1_C = 1_{(-n,n)},$$

$$C\subset(\otimes_{-n}^{n}(A_1,e_-,e_+))',$$

$$C\cong M_2\oplus M_3,$$

$$\|[U_\sigma 1_{(-n,n-1)},x]\|<\varepsilon\|x\| \quad \textit{for } x\in C.$$

Proof. We use the previous lemma for a sufficiently small $\varepsilon > 0$, where in the conclusion we may replace A by A_k for a sufficiently large k. By changing indices we may assume that $k = 2$.

Let $\ell \in \mathbf{N}$ be such that $\ell \gg 1$ and $\ell < (3\varepsilon)^{-1}$. By using the Rohlin property for σ on $\otimes_{\mathbf{Z}} A \cap A_2'$, we find an orthogonal family $(f_i)_{i=0}^{\ell-1}$ of projections and a unitary v in $\otimes_{\mathbf{Z}} A \cap A_2'$ such that $v \cong 1$, $\mathrm{Ad}\, vU_\sigma(f_i) = f_{i+1}$, and $[f_i] > (1+\ell)^{-1}[1]$ in $\mathbf{R} \otimes K_0(\otimes_{\mathbf{Z}} A \cap A_2')$. Let (e_{ij}) be a set of matrix units in C. We may assume that $[1-e] \leqslant [e_{11}ef_1]$ in $K_0(\otimes_{\mathbf{Z}} A \cap A_1')$ and let $b \in \otimes_{\mathbf{Z}} A \cap A_1'$ be a partial isometry such that $b^*b = 1 - e$ and $bb^* \leqslant e_{11}ef_1$. We define

$$y = \frac{1}{\sqrt{\ell}} \sum_{i=0}^{\ell-1} (vU_\sigma)^i b (vU_\sigma)^{-i}(1-F).$$

Then y belongs to $(\otimes_{\mathbf{Z}} A \cap A_1')(1-F) \subset (\otimes_{-n}^{n} A_1)'$, is close to a partial isometry

$$y_1 = \frac{1}{\sqrt{\ell}} \sum_{i=0}^{\ell-1} (vU_\sigma)^i b (VU_\sigma)^{-i}(1-F)$$

with $y_1^* y_1 = 1 - F$, and is close to

$$y_2 = \frac{1}{\sqrt{\ell}} \sum_{i=0}^{\ell-1} (VU_\sigma)^i b (vU_\sigma)^{-i}(1-F)$$

with $y_2 = Fe_{11}y_2$. Note also that

$$||\mathrm{Ad}\, vU_\sigma(y) - y|| \leqslant 2/\sqrt{\ell}.$$

Let v_1 be the partial isometry obtained by the polar decomposition of $Fe_{11}y1_{(-n,n)} = 1_{(-n,n)}Fe_{11}y \in (\otimes_{-n}^{n} A_1)'$, which is close to a partial isometry. Then it follows that $v_1^* v_1 = 1_{(-n,n)}(1-F)$, $v_1 v_1^* \leqslant 1_{(-n,n)}Fe_{11}$, and $[U, v_1] \sim 0$ for $U \equiv U_\sigma 1_{(-n,n-1)} = 1_{(-n+1,n)}U_\sigma$. Then the C^*-subalgebra C_1 generated by $C1_{(-n,n)}F$ and v_1 satisfies that $C_1 \subset (\otimes_{-n}^{n} A_1)'$ and $1_{C_1} = 1_{(-n,n)}$, $C_1 \cong M_2 \oplus M_3$. Moreover it follows that $||[U, x]||$ is small for x in the unit ball of C_1. Since $\otimes_{-n}^{n}(A_1, e_-, e_+) = (\otimes_{-n}^{n} A_1)1_{(-n,n)}$, this concludes the proof. $\square$

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